

Fluctuation Effect in the π -Flux State for Undoped High-Temperature Superconductors

Takao Morinari *

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

The effect of fluctuations about the π -flux mean field state for the undoped high-temperature superconductors is investigated. It is shown that fluctuations of the mean fields lead to a self-energy correction that doubles the band width of the fermion dispersion in the lowest order. The dynamical mass generation is associated with the self-energy effect due to the interaction mediated by the Lagrange multiplier field, which is introduced to impose the constraint on the fermions. A self-consistent picture about the mass generation and the propagation of the Lagrange multiplier field without damping is proposed. The antiferromagnetic long-range ordering is described without introducing an additional repulsive interaction. The theory suggests a natural framework to study spin disordered systems in which fermionic excitations are low-lying excitations.

KEYWORDS: high-temperature superconductivity, π -flux state, antiferromagnetic Heisenberg model, dynamical mass generation, confining potential

1. Introduction

In the phase diagram of the high-temperature superconductors, apparently the most established phases are the d -wave superconducting phase and the Néel ordering phase in the undoped compound. One of the key questions about high-temperature superconductivity is how to connect these two phases. If one starts from the Néel ordering state toward the d -wave superconducting state, the first step would be to consider the single hole doped system.

Experimentally such the system has been studied by angle-resolved photoemission spectroscopy (ARPES) in the undoped compounds¹⁻³ where a photo-hole is introduced in the system and the excitation spectrum is associated with the properties of the single hole doped system. The experimentally obtained excitation spectrum is in qualitatively good agreement with the quasiparticle excitation spectrum of the π -flux mean field state^{4,5} with a mass term.⁶ Although at the mean field level the quasiparticles are gapless, an excitation energy gap opens up by adding an on-site repulsive interaction. The gap arises from the staggered magnetization, and by taking it as a variational parameter, better variational energy is obtained.⁶ An effective field theory approach also suggests the presence of the mass term.⁷ In addition, the dispersions along $(0,0)$ - (π,π) line and $(0,\pi)$ - $(\pi,0)$ line in the Brillouin zone are isotropic as observed in the experiments.³ (The lattice constant is taken as the unit of length throughout

*E-mail: morinari@yukawa.kyoto-u.ac.jp

the paper.) However, the band width of the quasiparticles in the π -flux mean field state is smaller than the experiment by a factor of 2 – 3. Furthermore, there is no reliable estimation of the mass value except for variational wave function approaches. However, in the variational wave function approaches it is necessary to include an additional repulsive interaction.

On the other hand, self-consistent Born approximation analysis of the t-J model⁸ suggests that the band dispersion is significantly renormalized from $t \simeq 0.4\text{eV}$ to $J \simeq 0.13\text{eV}$, which is consistent with the experiments. However, there is some discrepancy in the excitation spectrum along the line in the momentum space from $(\pi, 0)$ to $(0, \pi)$. This discrepancy is removed by including the next-nearest and the third nearest neighbor hopping processes.⁹ In this approach, the t-J model is analyzed in terms of the slave-fermion mean field theory. In the slave-fermion mean field theory, the spins are described by the Schwinger bosons.¹⁰ Therefore, it is straightforward to describe the Néel ordering state as Bose-Einstein condensate of those bosons. However, to describe the d-wave superconducting state the slave-fermion formalism is not convenient. For the description of the d-wave superconducting state, slave-boson formalism is used in the literature.¹¹ In order to avoid taking a different formalism, here I focus on the π -flux state.

Before going into discussions about the π -flux state, let us discuss advantages and disadvantages of the theory. In order to consider the single hole starting from the spin 1/2 antiferromagnetic Heisenberg model, mainly there are two approaches: One is to represent spins in terms of boson fields, like in the non-linear σ model¹² or equivalently in the CP^1 model¹³ derived from the Schwinger boson mean field theory.¹⁰ In this approach, we need to think of a soliton-like excitation to describe a doped hole, which is fermion, in terms of boson fields. The situation is very similar to the Skyrme model¹⁴ in which fermions (protons and neutrons) are described as solitons in the non-linear σ model. (Goldstone modes, which correspond to spin wave excitations in the Heisenberg antiferromagnet, are pion fields.) It is argued that in ref. 15 that a doped hole can be described by a skyrmion-like spin texture.^{16–25} The band width and the mass, which is the excitation energy of the spin texture, are in good agreement with experimentally estimated values. (In the experiments, the mass value can be estimated by approximating the dispersion around $(\pm\pi/2, \pm\pi/2)$ by a conventional non-relativistic kinetic energy form.) From the Ginzburg-Landau description of the Heisenberg antiferromagnet, it is natural to expect the appearance of such a spin texture in the continuum: In the bosonic theory the Néel ordering state is described by a Bose-Einstein condensate. It turns out that local suppression of the order parameter, which is introduced by the formation of the Zhang-Rice singlet,²⁶ leads to a vortex-like state. This solution is easily found from the analysis of two-dimensional Ginzburg-Landau theory.²⁷ From the CP^1 formalism,²⁸ the vortex like state turns out to be a skyrmion like spin texture. However, stability of such a state on the lattice is not evident. This is because that the bosonic theory describes mainly the low-lying

excitations, that is, the spin wave excitations. Therefore, it is not easy to study microscopic stability of the spin texture within a bosonic theory.

By contrast, the other approach is to describe spins in terms of fermions:

$$S_{j\alpha} = \frac{1}{2} f_j^\dagger \sigma_\alpha f_j, \quad (1)$$

where $f_j^\dagger = (f_{j\uparrow}^\dagger, f_{j\downarrow}^\dagger)$, σ_α ($\alpha = x, y, z$) are the Pauli matrices. Because the spins are 1/2, there is a constraint on the fermions,

$$\sum_{\sigma=\uparrow,\downarrow} f_{j\sigma}^\dagger f_{j\sigma} = 1. \quad (2)$$

It is expected that the quantum nature of the spin 1/2 is well described by this theory. However, there is disadvantage in the description of the antiferromagnetic long-range ordering. Within the mean field theory, the staggered magnetization vanishes. Therefore, we need to go beyond the mean field theory to describe Néel ordering. In the continuum limit, the Néel ordering is discussed⁷ in the context of the dynamical mass generation in quantum electrodynamics in three spatial and time dimensions.^{29–33} Although there are discussions about the condition of the mass generation in the sense of the 1/N expansion, it is hard to estimate the value of the dynamically generated mass. Another way to describe the mass generation is to include the short-range Coulomb repulsion.⁶ However, it is not clear whether adding the short-range Coulomb repulsion term to the antiferromagnetic Heisenberg model is necessary. The antiferromagnetic Heisenberg model is derived from the Hubbard model taking the limit in which the on-site Coulomb repulsion, U , is much larger than the hopping matrix element, t . The condition of $U/t \gg 1$, is replaced by the constraint. So, it is unclear whether an additional repulsive interaction is necessary.

In this paper, we study the fluctuation effects in the π -flux state. The constraint is included in the action of the system using a standard Lagrange multiplier field. The effect of the constraint is studied by analyzing the self-energy effect associated with fluctuations of the Lagrange multiplier fields. It is argued that by including fluctuations the resulting quasiparticle band width and the staggered magnetization are in good agreement with the experiment. The result suggests that the π -flux state is a promising candidate for the description of the undoped high-temperature superconductors. In particular, the fact that the model described by the fermion fields includes the Néel ordering state implies that the theory provides us a natural framework to study doping effect and the spin disordered phase in the presence of the doped holes. Because the quantum nature of the holes are well described by such the theory.

The organization of this paper is as follows. In Sec.2 we study the fluctuations of the π -flux state mean fields. It is shown that the lowest order self-energy correction doubles the amplitude of the mean fields. In Sec.3 the coherent state path-integral action is introduced as a systematic approach. The effect of the Lagrange multiplier field and the π -flux state mean fields is studied in Sec.4, and the dynamical mass generation is discussed. In Sec.5 the spin

wave excitation, which is associated with phase fluctuations of the π -flux state mean fields, is discussed. Finally, section 6 is devoted to summary and discussion.

2. The π -Flux State and Perturbative Analysis of Fluctuations

In order to investigate the effect of fluctuations, let us start with the Hamiltonian of the π -flux state. Here we derive the mean field Hamiltonian and the term that describes fluctuations about the mean field state. The spin 1/2 antiferromagnetic Heisenberg model on the square lattice is given by

$$H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \quad (3)$$

where J is the superexchange interaction between the spins. The summation is taken over the nearest neighbor sites. As described in Introduction, the spins are represented by the fermions defined by eq.(1) with the constraint (2). In terms of these fermions, the Hamiltonian reads,

$$H = -\frac{1}{2}J \sum_{\langle i,j \rangle} \sum_{\alpha,\beta} f_{i\alpha}^\dagger f_{j\alpha} f_{j\beta}^\dagger f_{i\beta} + \sum_j \lambda_j \left(\sum_{\sigma} f_{j\sigma}^\dagger f_{j\sigma} - 1 \right), \quad (4)$$

up to a constant term. Here the constraint (2) is included by using a Lagrange multiplier. Now we introduce the following mean fields,

$$\chi_{j,j+\delta} = \sum_{\sigma} \langle f_{j+\delta,\sigma}^\dagger f_{j\sigma} \rangle, \quad (5)$$

where $\delta = \pm x, \pm y$. The π -flux state is obtained by assuming

$$\chi_{j,j+\hat{x}} = \chi_{j,j-\hat{x}} = \chi, \quad (6)$$

$$\chi_{j,j+\hat{y}} = \chi_{j,j-\hat{y}} = i\chi, \quad (7)$$

and $\lambda_j = 0$. After the Fourier transform the mean field Hamiltonian is given by

$$\begin{aligned} H_{MF} &= \sum_{k \in RBZ} \sum_{\alpha} \begin{pmatrix} f_{ek,\alpha}^\dagger & f_{ok,\alpha}^\dagger \end{pmatrix} \begin{pmatrix} 0 & -\kappa_k \\ -\kappa_k^* & 0 \end{pmatrix} \\ &\times \begin{pmatrix} f_{ek\alpha} \\ f_{ok\alpha} \end{pmatrix} + NJ|\chi|^2, \end{aligned} \quad (8)$$

where the summation with respect to k is taken over the reduced Brillouin zone, $|k_x - k_y| < \pi$ and $|k_x + k_y| < \pi$, and

$$\kappa_k = \chi J (\cos k_x + i \cos k_y). \quad (9)$$

The fields $f_{ek\sigma}$ and $f_{ok\sigma}$ are defined as

$$f_{e(o)k\sigma} = \sqrt{\frac{2}{N}} \sum_{j \in A(B)} f_{j\sigma} e^{-i\mathbf{k} \cdot \mathbf{R}_j}, \quad (10)$$

where the two sublattices are labeled by A and B . The mean field equation is given by

$$1 = \frac{J}{N} \sum_{k \in RBZ} \frac{\cos^2 k_x + \cos^2 k_y}{E_k} \tanh \frac{\beta E_k}{2}. \quad (11)$$

At $T = 0$, we find $\chi \simeq 0.479$. As discussed in Introduction, the fermion dispersion band width using this mean field value is half of the experimentally estimated value.

In order to calculate the self-energy correction associated with fluctuations, we study the Green's function. Introducing two component spinor,

$$f_{k\sigma}^\dagger = \begin{pmatrix} f_{ek\sigma}^\dagger & f_{ok\sigma}^\dagger \end{pmatrix}, \quad (12)$$

we define the Matsubara Green's function as follows,

$$G_{k\sigma}(\tau) = -\langle f_{k\sigma}(\tau) f_{k\sigma}^\dagger(0) \rangle. \quad (13)$$

In the mean field state the Green's function is

$$G_{k\uparrow}^{(0)}(i\omega_n) = \begin{pmatrix} \frac{u_k^2}{i\omega_n + E_k} + \frac{|v_k|^2}{i\omega_n - E_k} & u_k v_k \left(\frac{1}{i\omega_n + E_k} - \frac{1}{i\omega_n - E_k} \right) \\ u_k v_k^* \left(\frac{1}{i\omega_n + E_k} - \frac{1}{i\omega_n - E_k} \right) & \frac{|v_k|^2}{i\omega_n + E_k} + \frac{u_k^2}{i\omega_n - E_k} \end{pmatrix}, \quad (14)$$

$$G_{k\downarrow}^{(0)}(i\omega_n) = \begin{pmatrix} \frac{u_k^2}{i\omega_n + E_k} + \frac{|v_k|^2}{i\omega_n - E_k} & -u_k v_k \left(\frac{1}{i\omega_n + E_k} - \frac{1}{i\omega_n - E_k} \right) \\ -u_k v_k^* \left(\frac{1}{i\omega_n + E_k} - \frac{1}{i\omega_n - E_k} \right) & \frac{|v_k|^2}{i\omega_n + E_k} + \frac{u_k^2}{i\omega_n - E_k} \end{pmatrix}, \quad (15)$$

where $u_k = 1/\sqrt{2}$ and $v_k = \kappa_k/(\sqrt{2}|\kappa_k|)$.

Having defined the mean field Green's function, let us derive the term describing fluctuations. Fluctuations about the mean field, χ , arise from

$$\begin{aligned} H_{\text{int}} &= \frac{4J}{N} \sum_{k, k', q \in RBZ, q \neq 0} (\cos q_x + \cos q_y) \\ &\quad \times f_{o, k+q, \alpha}^\dagger f_{ok\beta} f_{ek'\beta}^\dagger f_{ek'+q\alpha}. \end{aligned} \quad (16)$$

Note that $q = 0$ contribution is excluded. Because the $q = 0$ term is included in the mean field Hamiltonian. In order to make the notation simpler, we omit $q \neq 0$ in the following equations. The interaction vertices are represented in Fig. 1. For example, the diagram with $a = \tau_o\sigma_+$ and $b = \tau_e\sigma_-$ in Fig.1 corresponds to the following term,

$$\begin{aligned} H_{(\tau_o\sigma_+, \tau_e\sigma_-)} &= \frac{4J}{N} \sum_{k, k', q \in RBZ} (\cos q_x + \cos q_y) \\ &\quad \times f_{o, k+q, \uparrow}^\dagger f_{ok\downarrow} f_{ek'\downarrow}^\dagger f_{ek'+q\uparrow}. \end{aligned} \quad (17)$$

The importance of including the fluctuation effect is clearly demonstrated by the lowest order self-energy correction. The first order self-energy with respect to H_{int} for the spin-up fermions is

$$\begin{aligned} \Sigma_{k\uparrow}(i\omega_n) &= -\frac{1}{\beta N} \sum_{q \in RBZ} \sum_{i\Omega_n} \sum_{\sigma} 2J_q \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} G_{k+q\sigma}(i\omega_n + i\Omega_n) \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \\ &\quad - \frac{1}{\beta N} \sum_{q \in RBZ} \sum_{i\Omega_n} \sum_{\sigma} 2J_q \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} G_{k+q\sigma}(i\omega_n + i\Omega_n) \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}. \end{aligned} \quad (18)$$

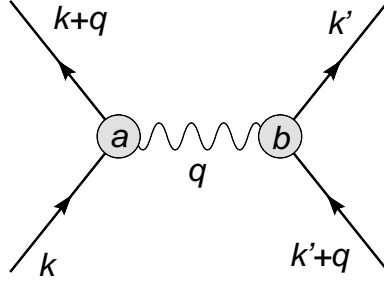


Fig. 1. The Feynman diagram for the interaction term associated with fluctuations of $\chi_{j,j+\delta}$. Here $(a, b) = (\tau_o\sigma_\uparrow, \tau_e\sigma_\uparrow), (\tau_o\sigma_\downarrow, \tau_e\sigma_\downarrow), (\tau_o\sigma_-, \tau_e\sigma_+), (\tau_o\sigma_+, \tau_e\sigma_-)$.

Straightforward calculation leads to

$$\begin{aligned} \Sigma_{k\uparrow}(i\omega_n) &= -\frac{2}{N} \sum_{k' \in RBZ} J_{k'-k} \begin{pmatrix} 0 & 2u_{k'}v_{k'} \\ 2u_{k'}v_{k'}^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -CJ \cos k_x - iCJ \cos k_y \\ -CJ \cos k_x + iCJ \cos k_y & 0 \end{pmatrix}, \end{aligned} \quad (19)$$

with $C \simeq 0.479$. Thus, the band width of the quasiparticle dispersion, which is given by $E_k^{MF} = \chi J \sqrt{\cos^2 k_x + \cos^2 k_y}$, is exactly doubled by this self-energy. Therefore, the quantitative difference of the quasiparticle band width compared to the experiment is much improved by including the fluctuation effect.

3. Coherent State Path-Integral Formalism

In principle, one can extend perturbative analysis in the previous section to higher order terms. However, since there are eight vertices, drawing Feynman diagrams is complicated. Furthermore, the Hamiltonian formalism is not useful for the study of fluctuations of the Lagrange multiplier field. For this purpose, the coherent state path-integral formulation is convenient. In the coherent state path-integral formulation, the partition function is given by

$$\Xi = \int Df^\dagger Df D\lambda \exp[-S], \quad (20)$$

where the action is

$$S = \int_0^\beta d\tau \left[\sum_j f_{j\sigma}^\dagger (\partial_\tau + \lambda_j) f_{j\sigma} - \frac{1}{2} J \sum_{\langle i,j \rangle} f_{i\alpha}^\dagger f_{j\alpha} f_{j\beta}^\dagger f_{i\beta} \right], \quad (21)$$

with $\beta = 1/k_B T$ the inverse temperature. After the Fourier transform, the Lagrangian is given by

$$\begin{aligned} L = & \sum_{k \in RBZ} \left(f_{ek\sigma}^\dagger \partial_\tau f_{ek\sigma} + f_{ok\sigma}^\dagger \partial_\tau f_{ok\sigma} \right) - \frac{J}{N} \sum_{k,k',q \in RBZ} \sum_{\delta=\pm x, \pm y} f_{ok+q\alpha}^\dagger f_{ek\alpha} f_{ek'\beta}^\dagger f_{ok'+q\beta} e^{-i(k-k')\cdot\delta} \\ & + \sum_{k,q \in RBZ} \left(\lambda_q f_{ek+q\sigma}^\dagger f_{ek\sigma} + \lambda_{q+Q} f_{ek+q\sigma}^\dagger f_{ek\sigma} + \lambda_q f_{ok+q\sigma}^\dagger f_{ok\sigma} - \lambda_{q+Q} f_{ok+q\sigma}^\dagger f_{ok\sigma} \right), \end{aligned} \quad (22)$$

where $Q = (\pi, \pi)$.

Now we introduce a Storaonovich-Hubbard transformation to rewrite the interaction term as follows,

$$\begin{aligned}
L = & \sum_{k \in RBZ} \left(f_{ek\sigma}^\dagger \partial_\tau f_{ek\sigma} + f_{ok\sigma}^\dagger \partial_\tau f_{ok\sigma} \right) \\
& + \sum_{q \in RBZ} \sum_{k \in RBZ} \left(\lambda_q f_{ek+q\sigma}^\dagger f_{ek\sigma} + \lambda_{q+Q} f_{ek+q\sigma}^\dagger f_{ek\sigma} + \lambda_q f_{ok+q\sigma}^\dagger f_{ok\sigma} - \lambda_{q+Q} f_{ok+q\sigma}^\dagger f_{ok\sigma} \right) \\
& - \frac{J}{2} \sum_{q \in RBZ} \sum_{\delta=\pm x, \pm y} \chi_q^{(\delta)*} \left(\sqrt{\frac{2}{N}} \sum_{k \in RBZ} e^{ik \cdot \delta} f_{ek\alpha}^\dagger f_{ok+q\alpha} \right) \\
& - \frac{J}{2} \sum_{q \in RBZ} \sum_{\delta=\pm x, \pm y} \chi_q^{(\delta)} \left(\sqrt{\frac{2}{N}} \sum_{k \in RBZ} e^{-ik \cdot \delta} f_{ok+q\alpha}^\dagger f_{ek\alpha} \right) \\
& + \frac{J}{2} \sum_{q \in RBZ} \sum_{\delta=\pm x, \pm y} \chi_q^{(\delta)*} \left(\chi_q^{(\delta)} \right)
\end{aligned}$$

At the saddle point, we see that

$$\chi_q^{(\delta)} = \sqrt{\frac{2}{N}} \sum_{k \in RBZ} e^{ik \cdot \delta} \langle f_{ek\alpha}^\dagger f_{ok+q\alpha} \rangle. \quad (23)$$

As a mean field state, we assume that

$$\begin{aligned}
\chi_q^{(+x)} &= \sqrt{\frac{N}{2}} \chi_1 \delta_{q,0}, & \chi_q^{(+y)} &= \sqrt{\frac{N}{2}} \chi_2 \delta_{q,0}, \\
\chi_q^{(-x)} &= \sqrt{\frac{N}{2}} \chi_3 \delta_{q,0}, & \chi_q^{(-y)} &= \sqrt{\frac{N}{2}} \chi_4 \delta_{q,0}, \\
\lambda_q &= 0, & \lambda_{q+Q} &= 0.
\end{aligned} \quad (24)$$

The action is

$$\begin{aligned}
S = & \int_0^\beta d\tau \left[\sum_{k \in RBZ} \begin{pmatrix} f_{ek\alpha}^\dagger & f_{ok\alpha}^\dagger \end{pmatrix} \begin{pmatrix} \partial_\tau & \kappa_k^* \\ \kappa_k & \partial_\tau \end{pmatrix} \begin{pmatrix} f_{ek\alpha} \\ f_{ok\alpha} \end{pmatrix} \right. \\
& \left. + \frac{NJ}{4} \left(|\chi_1|^2 + |\chi_2|^2 + |\chi_3|^2 + |\chi_4|^2 \right) \right], \quad (25)
\end{aligned}$$

where

$$\kappa_k = -\frac{J}{2} \left(\chi_1 e^{-ik_x} + \chi_2 e^{-ik_y} + \chi_3 e^{ik_x} + \chi_4 e^{ik_y} \right). \quad (26)$$

For the π -flux state, χ_j 's are

$$\chi_1 = \chi_3 = -i\chi_2 = -i\chi_4 \equiv \chi. \quad (27)$$

The saddle point equation, or the mean field equation, is given by eq. (11).

4. Effect of Fluctuations

Now we study the fluctuations of the Lagrange multiplier field and the π -flux state mean fields, χ . Fluctuations about the saddle point are described by the following action,

$$S_f = \int_0^\beta d\tau \left[\frac{J}{2} \sum_{q \in RBZ} \sum_{\delta=\pm x, \pm y} |\chi_{q\delta}|^2 + \sum_{k, q \in RBZ} \begin{pmatrix} f_{ek+q\sigma}^\dagger & f_{ok+q\sigma}^\dagger \end{pmatrix} \right. \\ \left. \times \begin{pmatrix} \lambda_{eq} & -\frac{J}{\sqrt{2N}} \sum_{\delta=\pm x, \pm y} e^{i(k+\frac{q}{2})\cdot\delta} \chi_{-q, \delta}^* \\ -\frac{J}{\sqrt{2N}} \sum_{\delta=\pm x, \pm y} e^{-i(k+\frac{q}{2})\cdot\delta} \chi_{q, \delta} & \lambda_{oq} \end{pmatrix} \begin{pmatrix} f_{ek\sigma} \\ f_{ok\sigma} \end{pmatrix} \right] \quad (28)$$

where $\chi_{q, \delta} = e^{iq\cdot\delta/2} \chi_q^{(\delta)}$ and $\lambda_{eq} = \lambda_q + \lambda_{q+Q}$ and $\lambda_{oq} = \lambda_q - \lambda_{q+Q}$. We consider fluctuations within the Gaussian approximation. At this point, we include the staggered magnetization term:

$$S_{st} = \int_0^\beta d\tau \sum_{k \in RBZ} \begin{pmatrix} f_{ek}^\dagger & f_{ok}^\dagger \end{pmatrix} \begin{pmatrix} -\Delta_{st}\sigma_z & 0 \\ 0 & \Delta_{st}\sigma_z \end{pmatrix} \begin{pmatrix} f_{ek} \\ f_{ok} \end{pmatrix}. \quad (29)$$

Within the mean field theory, $\Delta_{st} = 0$. Nonzero value of Δ_{st} arises from the self-energy effect as shall be seen below. Integrating out the fermion fields leads to the following effective action,

$$S_f = -\frac{1}{8} \sum_q \begin{pmatrix} \lambda_{eq} & \lambda_{oq} \end{pmatrix} \begin{pmatrix} \Pi_0^\lambda(q) & -\Pi_1^\lambda(q) \\ -\Pi_1^\lambda(q) & \Pi_0^\lambda(q) \end{pmatrix} \begin{pmatrix} \lambda_{e,-q} \\ \lambda_{o,-q} \end{pmatrix} \\ + \frac{J}{2} \sum_q \sum_{\delta=\pm x, \pm y} \begin{pmatrix} \chi_{q\delta}^* & \chi_{-q, \delta} \end{pmatrix} \begin{pmatrix} 1 - \Pi_0^\chi(q) & \Pi_1^\chi(q) \\ \Pi_2^\chi(q) & 1 - \Pi_0^\chi(q) \end{pmatrix} \begin{pmatrix} \chi_{q\delta} \\ \chi_{-q, \delta}^* \end{pmatrix}, \quad (30)$$

where

$$\Pi_0^\lambda(q) = \sum_{\mathbf{k} \in RBZ} \left(\frac{1}{i\Omega_n + E_{\mathbf{k}+\mathbf{q}} + E_{\mathbf{k}}} - \frac{1}{i\Omega_n - E_{\mathbf{k}+\mathbf{q}} - E_{\mathbf{k}}} \right), \quad (31)$$

$$\Pi_1^\lambda(q) = \sum_{\mathbf{k} \in RBZ} \frac{\kappa_{\mathbf{k}}^* \kappa_{\mathbf{k}+\mathbf{q}} + \Delta_{st}^2}{E_{\mathbf{k}} E_{\mathbf{k}+\mathbf{q}}} \left(\frac{1}{i\Omega_n + E_{\mathbf{k}+\mathbf{q}} + E_{\mathbf{k}}} - \frac{1}{i\Omega_n - E_{\mathbf{k}+\mathbf{q}} - E_{\mathbf{k}}} \right), \quad (32)$$

$$\Pi_0^\chi(q) = \frac{J}{8N} \sum_{\mathbf{k} \in RBZ} \left(\frac{1}{i\Omega_n + E_{\mathbf{k}+\mathbf{q}} + E_{\mathbf{k}}} - \frac{1}{i\Omega_n - E_{\mathbf{k}+\mathbf{q}} - E_{\mathbf{k}}} \right), \quad (33)$$

$$\Pi_1^\chi(q) = \frac{J}{N} \sum_{\mathbf{k} \in RBZ} \frac{\kappa_{\mathbf{k}} \kappa_{\mathbf{k}+\mathbf{q}} - \Delta_{st}^2}{E_{\mathbf{k}} E_{\mathbf{k}+\mathbf{q}}} \left(\frac{1}{i\Omega_n + E_{\mathbf{k}+\mathbf{q}} + E_{\mathbf{k}}} - \frac{1}{i\Omega_n - E_{\mathbf{k}+\mathbf{q}} - E_{\mathbf{k}}} \right), \quad (34)$$

$$\Pi_2^\chi(q) = \frac{J}{N} \sum_{\mathbf{k} \in RBZ} \frac{\kappa_{\mathbf{k}}^* \kappa_{\mathbf{k}+\mathbf{q}}^* - \Delta_{st}^2}{E_{\mathbf{k}} E_{\mathbf{k}+\mathbf{q}}} \left(\frac{1}{i\Omega_n + E_{\mathbf{k}+\mathbf{q}} + E_{\mathbf{k}}} - \frac{1}{i\Omega_n - E_{\mathbf{k}+\mathbf{q}} - E_{\mathbf{k}}} \right). \quad (35)$$

Here $E_{\mathbf{k}} = \sqrt{\chi^2 J^2 (\cos^2 k_x + \cos^2 k_y) + \Delta_{st}^2}$, and q denotes $(\mathbf{q}, i\Omega_n)$ with $\mathbf{q} \in RBZ$ and $\Omega_n = 2\pi n/\beta$ being the bosonic Matsubara frequency.

In order to make clear the physical meaning of λ_q and $\chi_{q\delta}$, we study them separately. First, let us investigate the effect of the former. As we will see below, exchange of λ_q fields

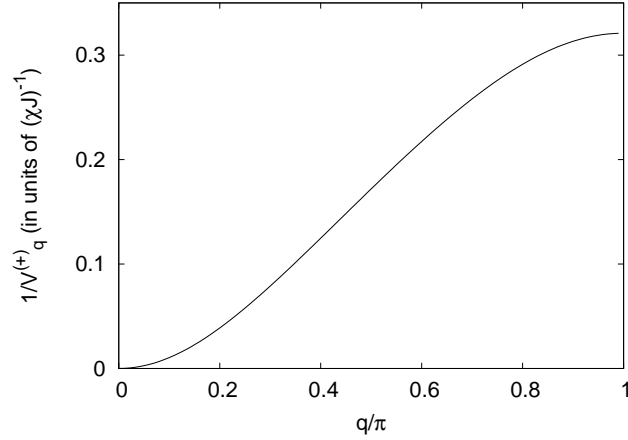


Fig. 2. The inverse of the potential $V_{\mathbf{q}=(q,q)}^{(+)}$ versus q/π .

leads to a logarithmic confining potential between the fermions in the presence of the fermion mass term. The coupling term between λ_q and the fermions is

$$\begin{aligned}
 S_{\lambda}^{\text{int}} = & \sum_{k,q} \left[\frac{1}{2} \begin{pmatrix} \lambda_{eq} & \lambda_{oq} \end{pmatrix} \begin{pmatrix} f_{e\mathbf{k}+\mathbf{q}\sigma}^{\dagger} f_{e\mathbf{k}\sigma} \\ f_{o\mathbf{k}+\mathbf{q}\sigma}^{\dagger} f_{o\mathbf{k}\sigma} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} f_{e\mathbf{k}\sigma}^{\dagger} f_{e\mathbf{k}+\mathbf{q}\sigma} & f_{o\mathbf{k}\sigma}^{\dagger} f_{o\mathbf{k}+\mathbf{q}\sigma} \end{pmatrix} \begin{pmatrix} \lambda_{e,-q} \\ \lambda_{o,-q} \end{pmatrix} \right] \\
 & - \frac{1}{8} \sum_q \begin{pmatrix} \lambda_{eq} & \lambda_{oq} \end{pmatrix} \begin{pmatrix} \Pi_0^{\lambda}(q) & -\Pi_1^{\lambda}(q) \\ -\Pi_1^{\lambda}(q) & \Pi_0^{\lambda}(q) \end{pmatrix} \begin{pmatrix} \lambda_{e,-q} \\ \lambda_{o,-q} \end{pmatrix}. \quad (36)
 \end{aligned}$$

Note that using this action in deriving the effective interaction term is equivalent to the random phase approximation. This point is demonstrated using a simple model in Appendix. Integrating out λ_q leads to the following interaction term,

$$\begin{aligned}
 S_{\lambda} = & \sum_q \begin{pmatrix} \rho_{eq} + \rho_{oq} & \rho_{eq} - \rho_{oq} \end{pmatrix} \begin{pmatrix} V_q^{(+)} & 0 \\ 0 & V_q^{(-)} \end{pmatrix} \\
 & \times \begin{pmatrix} \rho_{e,-q} + \rho_{o,-q} \\ \rho_{e,-q} - \rho_{o,-q} \end{pmatrix}, \quad (37)
 \end{aligned}$$

where

$$V_q^{(\pm)} = \frac{1}{\Pi_0^{\lambda}(q) \mp \Pi_1^{\lambda}(q)}. \quad (38)$$

One can see that for the static case, $i\Omega_n = 0$, the denominator of $V_q^{(+)}$ behaves as $1/q^2$ for $q \rightarrow 0$ as seen from the expansion with respect to q . ($1/V_q^{(+)}$ is shown in Fig.2.) Therefore, the potential term $V_q^{(+)}$ is a logarithmic potential.³⁴ Since such a potential diverges at long distance, it is a kind of confining potential. The same instantaneous interaction term is obtained in the continuum theory of QED₃ by taking the Coulomb gauge.

Now we examine dynamical mass generation associated with this logarithmic confining potential. We consider the first order self-energy term. The equation to determine the self-

energy in a self-consistent manner is

$$\Sigma_{e\mathbf{k}}(i\omega_n) = -\frac{1}{\beta N} \sum_{\mathbf{q}} V_{\mathbf{q}}^{(+)} G_{e\mathbf{k}+\mathbf{q}}(i\omega_n + i\Omega_n). \quad (39)$$

A similar equation holds for $\Sigma_{o\mathbf{k}}(i\omega_n)$. Here $G_{e\mathbf{k}}(i\omega_n) = G_{e\mathbf{k}}^{(0)}(i\omega_n) - \Sigma_{e\mathbf{k}}(i\omega_n)$. To study the dynamical mass generation, we need to consider a non-uniform mass term in general. However, for such a general mass term we are unable to obtain analytic forms for the propagators. To solve the equation (39) self-consistently we need to solve the coupled equations numerically. Here we discuss the dynamical mass generation within the approximation of the uniform mass term, $\Sigma_{e\mathbf{k}}(i\omega_n) = -\Delta_{st}\sigma_z$. Under this approximation, we find the equation for the mass,

$$\frac{1}{N} \sum_{\mathbf{q}} \frac{V_{\mathbf{q}}^{(+)}}{2E_{(\pi/2, \pi/2)+\mathbf{q}}} = 1. \quad (40)$$

In spite of this simple form, numerically solving this gap equation is not an easy task. Because the potential is singular, it is necessary to introduce an infrared cutoff and the result depends on it. Furthermore, the result also depends on the number of lattice points introduced for numerical estimations.³³ Instead of precisely determining the mass gap value, we evaluate it approximately. After some algebra, we find

$$\Pi_{ee}^{\lambda}(q) - \Pi_{eo}^{\lambda}(q) = \frac{1}{N} \sum_{\mathbf{k} \in RBZ} \frac{A_{\mathbf{k},\mathbf{q}}^2}{E_{\mathbf{k}}^3} + \left(\sin^2 \frac{q_x}{2} + \sin^2 \frac{q_y}{2} \right) \frac{1}{N} \sum_{\mathbf{k} \in RBZ} \frac{(\chi J)^2}{E_{\mathbf{k}}^3}, \quad (41)$$

where

$$A_{\mathbf{k},\mathbf{q}}^2 \simeq \frac{1}{4} q_{\alpha} q_{\beta} \left(\frac{\partial^2 E_{\mathbf{k}}}{\partial k_{\alpha} \partial k_{\beta}} E_{\mathbf{k}} - \frac{\partial E_{\mathbf{k}}}{\partial k_{\alpha}} \frac{\partial E_{\mathbf{k}}}{\partial k_{\beta}} \right). \quad (42)$$

In the k -summation in eq.(41), dominant contribution comes from $\mathbf{k} \simeq (\pm\pi/2, \pm\pi/2)$. Taking the approximate form around these points, we obtain

$$\Pi_{ee}^{\lambda}(q) - \Pi_{eo}^{\lambda}(q) \simeq \frac{1}{2} \frac{(\chi J)^2}{\Delta_{st}^3} \mathbf{q}^2. \quad (43)$$

Now the gap equation is

$$\int_0^{\pi} \frac{dq}{2\pi} \frac{\Delta_{st}^2}{(\chi J)^2 q} \simeq 1. \quad (44)$$

Note that this equation is different from that for QED₃ by the factor of $(\Delta_{st}/\chi J)^2$ in which the potential term is not proportional to Δ_{st} . Apparently this gap equation is suffering from the infrared divergence. In order to properly deal with this infrared divergence, we need to take into account the vertex correction. In the QED₃ theory there is controversy in the choice of the vertex correction.^{35,36} Here we determine it from a physical argument. The interaction term in the gap equation comes from the following term,

$$H_{\text{int}} = \sum_{\mathbf{q}} V_{\mathbf{q}}^{(+)} \delta \rho_{\mathbf{q}} \delta \rho_{-\mathbf{q}}, \quad (45)$$

where $\delta\rho_q = \rho_q - \bar{\rho}$, with $\bar{\rho}$ being the uniformly distributed background particle density. Note that $\delta\rho_q \rightarrow 0$ for $q \rightarrow 0$. On the other hand, from the density-density correlation function, $\langle \delta\rho_q \delta\rho_{-q} \rangle \sim q^2$. This suggests that the approximate form of the vertex correction is

$$\gamma_q = \begin{cases} (\Delta_{st}/\chi J)^{-\alpha/2} q^\alpha & \left(\text{for } q < \sqrt{\Delta_{st}/\chi J} \right) \\ 1 & \left(\text{for } q > \sqrt{\Delta_{st}/\chi J} \right) \end{cases}, \quad (46)$$

with $\alpha = 1$. Dividing the integration with respect to q in eq.(44) with including γ_q , into $0 < q < \sqrt{(\Delta_{st}/\chi J)}$ and $q > \sqrt{(\Delta_{st}/\chi J)}$,

$$2\pi(\Delta_{st}/\chi J)^2 - \log\left(\frac{\pi}{(\Delta_{st}/\chi J)}\right) \simeq 1. \quad (47)$$

This equation can be solved numerically, and the solution is

$$\Delta_{st} \simeq 0.616\chi J. \quad (48)$$

Substituting $\chi \simeq 2 \times 0.479$, we find $\Delta_{st} \simeq 0.59J$. This value corresponds to the staggered magnetization of $m_{st} \simeq 0.30$, which is a reasonable value compared to the numerical simulation results and the experiments.

Having discussed the dynamical mass generation associated with the Lagrange multiplier field fluctuations, let us move on to the mean field fluctuations, $\chi_{q\delta}$. The fluctuation mode associated with $\chi_{q\delta}$ is found from the pole of the following equation,

$$\det \begin{pmatrix} 1 - \Pi_0^\chi(q) & \Pi_1^\chi(q) \\ \Pi_2^\chi(q) & 1 - \Pi_0^\chi(q) \end{pmatrix} = 0. \quad (49)$$

The pole was found from numerical computations with mesh size of 64×64 and taking $\delta = 0.10$, where δ is introduced in the analytic continuation from the Matsubara frequency, $i\Omega_n$ to the real frequency, through $i\Omega_n = \omega + i\delta$. It was found that the amplitude fluctuation mode has the excitation energy gaps whose minimum is about $1.2J$. Along the line from $(0, 0)$ to $(\pi/2, \pi/2)$ in the reduced Brillouin zone, the band minimum is located at $(0, 0)$, and the maximum is at $(\pi/2, \pi/2)$. The band width along this line is $1.3J$. The excitation energy at other points are within this energy range. Although the amplitude fluctuation mode of $\chi_{q\delta}$ is high-energy mode and can be neglected, the phase fluctuation of $\chi_{q\delta}$ is related to the spin-wave excitation. Because there is the relation,

$$\langle \mathbf{S}_j \cdot \mathbf{S}_{j+\delta} \rangle = |\chi_{j,j+\delta}|^2 = \left| \sqrt{\frac{2}{N}} \sum_{\mathbf{q} \in RBZ} e^{i\mathbf{q} \cdot (\mathbf{R}_j + \delta/2)} \chi_{q,\delta} \right|^2. \quad (50)$$

The spin wave excitation is discussed in the next section.

5. Spin-Wave Excitations

In this section, we consider the spin wave excitation associated with the phase fluctuations of the mean field, $\chi_{q\delta}$ in the presence of the dynamically generated mass, or the staggered

magnetization. The calculation is similar to the spin-density wave theory.^{37,38} Here we calculate the spin wave excitation spectrum following ref.39.

The repulsive interaction $V_q^{(+)}$ that leads to the non-zero staggered magnetization is approximated by a short-range repulsion, V , for simplicity. The interaction V is associated with a weak short-range repulsive interaction assumed by Hsu.⁶ The value of V is evaluated from

$$\frac{1}{N} \sum_{\mathbf{k} \in RBZ} \frac{1}{\sqrt{|\kappa_{\mathbf{k}}|^2 + |\Delta_{st}|^2}} = \frac{1}{V}.$$

Using the values of $\chi = 0.994$ and $\Delta_{st} = 0.616J$, we obtain $V = 2.2J$. The action that describes fluctuations is given by

$$\begin{aligned} S = & \int_0^\beta d\tau \sum_{\mathbf{k}} \begin{pmatrix} f_{\mathbf{k}}^\dagger & f_{\mathbf{k}+Q}^\dagger \end{pmatrix} \begin{pmatrix} \partial_\tau + \chi J \cos k_x & -i\chi J \cos k_y - \Delta_{st}\sigma_z \\ i\chi J \cos k_y - \Delta_{st}\sigma_z & \partial_\tau - \chi J \cos k_x \end{pmatrix} \begin{pmatrix} f_{\mathbf{k}} \\ f_{\mathbf{k}+Q} \end{pmatrix} \\ & + \int_0^\beta d\tau \left[- \sum_{\mathbf{k}, \mathbf{k}'} \delta_{\mathbf{m}_{\mathbf{k}-\mathbf{k}'}} \cdot (f_{\mathbf{k}}^\dagger \sigma f_{\mathbf{k}'}) + \frac{1}{V} \sum_{\mathbf{q}} \delta_{\mathbf{m}_{\mathbf{q}}} \cdot \delta_{\mathbf{m}_{-\mathbf{q}}} \right] + \frac{\beta N}{V} \Delta_{st}^2. \end{aligned} \quad (51)$$

Integrating out fermions, and after some algebra, we obtain

$$\begin{aligned} S_{eff} = & \frac{1}{V} \sum_{\mathbf{q} \in RBZ} (a_{\mathbf{q}} \delta m_{\mathbf{q}}^z \delta m_{-\mathbf{q}}^z + a_{\mathbf{q}+Q} \delta m_{\mathbf{q}+Q}^z \delta m_{-\mathbf{q}-Q}^z) \\ & + \frac{1}{V} \sum_{\mathbf{q} \in RBZ} \begin{pmatrix} \delta m_{\mathbf{q}}^x & \delta m_{\mathbf{q}+Q}^y \end{pmatrix} \begin{pmatrix} b_{\mathbf{q}} & c_{\mathbf{q}} \\ -c_{\mathbf{q}} & b_{\mathbf{q}+Q} \end{pmatrix} \begin{pmatrix} \delta m_{-\mathbf{q}}^x \\ \delta m_{-\mathbf{q}-Q}^y \end{pmatrix} \\ & + \frac{1}{V} \sum_{\mathbf{q} \in RBZ} \begin{pmatrix} \delta m_{\mathbf{q}+Q}^x & \delta m_{\mathbf{q}}^y \end{pmatrix} \begin{pmatrix} b_{\mathbf{q}} & -c_{\mathbf{q}} \\ c_{\mathbf{q}} & b_{\mathbf{q}+Q} \end{pmatrix} \begin{pmatrix} \delta m_{-\mathbf{q}-Q}^x \\ \delta m_{-\mathbf{q}}^y \end{pmatrix}, \end{aligned} \quad (52)$$

where

$$a_{\mathbf{q}} = 1 - \frac{1}{2}V \left[K_{\mathbf{q}}^{(0)} - K_{\mathbf{q}}^{(z)} \right], b_{\mathbf{q}} = 1 - \frac{1}{2}V K_{\mathbf{q}}^{(0)}, c_{\mathbf{q}} = \frac{1}{2}V K_{\mathbf{q}}^{(u)}, \quad (53)$$

with

$$K_{\mathbf{q}}^{(0)} = \frac{1}{N} \sum_{\mathbf{k} \in RBZ} \left(1 - \frac{\kappa_{\mathbf{k}+\mathbf{q}}^* \kappa_{\mathbf{k}} + \kappa_{\mathbf{k}+\mathbf{q}} \kappa_{\mathbf{k}}^* - 2\Delta_{st}^2}{2E_{\mathbf{k}+\mathbf{q}} E_{\mathbf{k}}} \right) \left(\frac{1}{i\Omega_n + E_{\mathbf{k}+\mathbf{q}} + E_{\mathbf{k}}} - \frac{1}{i\Omega_n - E_{\mathbf{k}+\mathbf{q}} - E_{\mathbf{k}}} \right), \quad (54)$$

$$K_{\mathbf{q}}^{(z)} = \frac{1}{N} \sum_{\mathbf{k} \in RBZ} \frac{2\Delta_{st}^2}{E_{\mathbf{k}+\mathbf{q}} E_{\mathbf{k}}} \left(\frac{1}{i\Omega_n + E_{\mathbf{k}+\mathbf{q}} + E_{\mathbf{k}}} - \frac{1}{i\Omega_n - E_{\mathbf{k}+\mathbf{q}} - E_{\mathbf{k}}} \right), \quad (55)$$

$$K_{\mathbf{q}}^{(u)} = \frac{1}{N} \sum_{\mathbf{k} \in RBZ} (-i\Delta_{st}) \left(\frac{1}{E_{\mathbf{k}}} + \frac{1}{E_{\mathbf{k}+\mathbf{q}}} \right) \left(\frac{1}{i\Omega_n + E_{\mathbf{k}+\mathbf{q}} + E_{\mathbf{k}}} + \frac{1}{i\Omega_n - E_{\mathbf{k}+\mathbf{q}} - E_{\mathbf{k}}} \right). \quad (56)$$

The spin wave excitation is associated with the pole of the transverse spin fluctuations. The pole is found from

$$\det \begin{pmatrix} b_{\mathbf{q}} & c_{\mathbf{q}} \\ -c_{\mathbf{q}} & b_{\mathbf{q}+Q} \end{pmatrix} = 0, \quad (57)$$

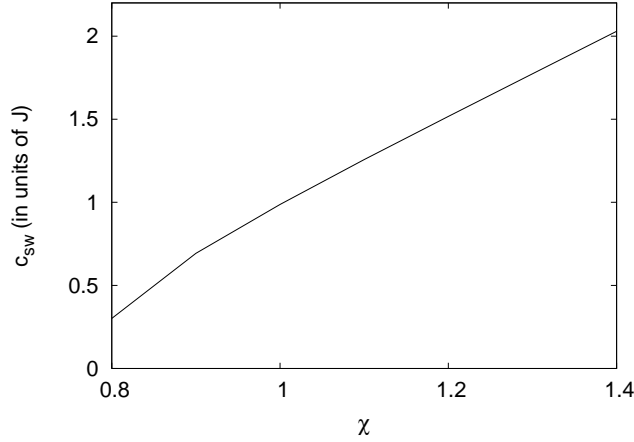


Fig. 3. The mean field χ dependence of the spin-wave velocity, c_{sw} .

that is,

$$\left(1 - \frac{1}{2}VK_q^{(0)}\right) \left(1 - \frac{1}{2}VK_{q+Q}^{(0)}\right) + \left(\frac{1}{2}VK_q^{(u)}\right)^2 = 0. \quad (58)$$

We perform the analytic continuation of $i\Omega_n \rightarrow \omega + i\delta$, and then expand each quantity with respect to ω and q . Noting that

$$\frac{1}{2}K_Q^{(0)}(i\Omega_n = 0) = \frac{1}{N} \sum_{k \in RBZ} \frac{1}{E_{\mathbf{k}}} = \frac{1}{V}, \quad (59)$$

we find, after some algebra and a numerical computation,

$$\omega \simeq 0.87Jq. \quad (60)$$

Thus, the spin-wave velocity is $c_{sw} = 0.87J$. This result is in agreement with the known established value of $c_{sw} = 1.65J$ estimated from the quantum Monte Carlo simulations and a series expansion.^{40–42} However, the value of c_{sw} depends on χ . Since the estimation of χ based on a perturbative analysis, the value of χ would change by further including higher order terms. Here we do not attempt to estimate χ precisely in this way because it is hard to fix the value from that procedure. Instead, we compute χ dependence of c_{sw} by fixing $\Delta_{st} = 0.60J$, which is the constraint from the experiments and the numerical simulations. In Fig.3, χ versus c_{sw} is shown. From the known value of c_{sw} , χ is evaluated instead. $c_{sw} \simeq 1.65$ is obtained by setting $\chi = 1.25$. This value implies that the quasiparticle band width is $1.76J$, which is a reasonable value compared to the experiment.

In the limit of $\Delta_{st}/(\chi J) \gg 1$, the spin wave dispersion is easily obtained⁶ as in the spin-density wave state.^{37,38} Using the approximation like

$$\frac{1}{E_{\mathbf{k}}} \simeq \frac{1}{\Delta_{st}} - \frac{|\kappa_{\mathbf{k}}|^2}{2\Delta_{st}^3},$$

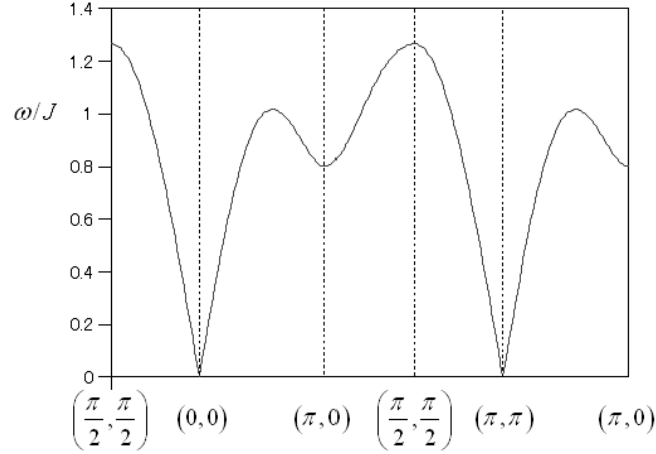


Fig. 4. Numerically calculated spin wave dispersion along high-symmetry directions. The lattice size is 40×40 and $\delta = 0.2$.

and noting

$$\frac{1}{N} \sum_{\mathbf{k} \in RBZ} \kappa_{\mathbf{k}+\mathbf{q}}^* \kappa_{\mathbf{k}} = \frac{(\chi J)^2}{4} (\cos q_x + \cos q_y), \quad (61)$$

we find the following dispersion,

$$\omega_q = \frac{(\chi J)^2}{\Delta_{st}} \sqrt{1 - \frac{1}{4} (\cos q_x + \cos q_y)^2}. \quad (62)$$

This coincides with the result of the spin wave theory of the antiferromagnetic Heisenberg model. The same form of the dispersion is also obtained in the spin-density wave state in the strong coupling limit.^{37,38} Although the assumption of $\Delta_{st}/(\chi J) \gg 1$ is not valid for excitations with $\omega > \Delta_{st}$, eq.(62) is better for high-energy excitations than numerically obtained spin wave dispersion⁶ shown in Fig.4 along high-symmetry directions in the Brillouin zone. The situation is similar to the spin density wave state as shown in Fig.5.

The full spin wave excitation spectrum is investigated by the neutron scattering experiments in the undoped compound.⁴³ Most of features is in good agreement with the spin wave dispersion (62) with a suitable prefactor except around $(\pi, 0)$. The dip around this point can be explained by including a ring exchange term to the Heisenberg model.^{43,44} Effect of ring-exchange interaction on the π -flux state is considered in ref.45 within the mean field theory. Although it would be interesting to investigate fluctuations about the mean field state including the ring-exchange interaction, consideration of such an effect is beyond the scope of this paper.

6. Summary and Discussion

In this paper, the effect of fluctuations about the π -flux mean field state has been investigated. As for the fluctuations of the Lagrange multiplier field, it is shown that the fluctuations

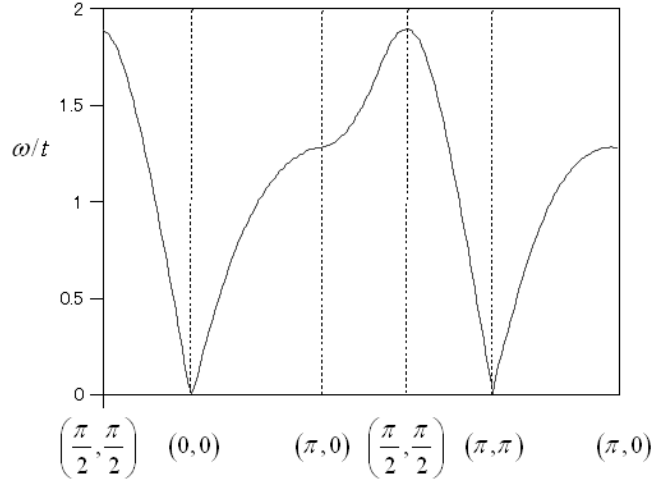


Fig. 5. Numerically calculated spin wave dispersion along high-symmetry directions for the spin-density wave state at half-filling. The lattice size is 40×40 and $\delta = 0.1$. The on-site Coulomb repulsion is $U = 5t$.

lead to a logarithmic potential term in the presence of the mass term. The mass value is evaluated by solving the Schwinger-Dyson equation approximately, and a reasonable value is obtained. Because of the mass term, there is an energy gap in particle-hole excitations. As a result, the interaction mediated by the Lagrange multiplier field fluctuations is long-ranged. If the fermions were massless, the interaction would be short-ranged. (For a different approach to the constraint, see ref.46.)

As for the fluctuations of the π -flux state mean fields, the lowest order self-energy correction doubles the mean field value. The resulting quasiparticle band width is much closer to that evaluated in the ARPES experiment¹ compared to the mean field result. From the pole of the correlation function, it is found that the amplitude fluctuation mode has a gap of $1.2J$. Meanwhile, the phase fluctuations are associated with the spin wave excitation, which is analyzed by approximating the potential term mediated by the Lagrange multiplier field fluctuations by a short-range repulsion.

The dynamical mass generation is consistent with the confinement of fermionic excitations. That is, there are no low-lying fermionic excitations in the two-dimensional Heisenberg antiferromagnet. Due to the mass term of the fermions on the order of J , the propagation of the Lagrange multiplier field do not excite particle-hole pairs. Therefore, there is no damping for the propagation. As a result, there is a long-range interaction between the fermions. Since the Lagrange multiplier field lives on three spatial and time dimensions, the interaction between fermions mediated by the exchange of the Lagrange multiplier fields is a logarithmic potential. Under the effect of such a confining potential, the fermions are confined. This is, of course, consistent with the constraint. Because the constraint is satisfied for low-energy exci-

tations. Contrary, if the fermions were massless, the propagation of the Lagrange multiplier fields would excite many particle-hole pairs. The interaction between fermions mediated by the exchange of such a field is short range interaction. Therefore, there would be no confinement, or fermionic excitations would appear in the low-lying excitations, and the constraint would be no longer satisfied. This picture suggests that a small mass term in the Lagrange multiplier field should break the confinement of the fermions. This contradicts with the conclusion in ref.47 in which it is argued that the dynamical mass generation occurs in the presence of a small gauge field mass.

As for the application to the high-temperature superconductors, the theory suggests that the low-lying excitations in the spin disordered regime is described by the fermions with the background of the π -flux state correlations.⁴⁸ Because in the spin disordered regime there is no staggered magnetization. Therefore, there is no mass gap for particle-hole excitations. In this case the interaction between the fermions mediated by the Lagrange multiplier fields is of short range interaction. We may neglect the effect of such a short-range interaction. Meanwhile, the phase fluctuations are important for the properties of the fermions. Because the phase fluctuations play the role of transverse gauge field living in three spatial and time dimensions. As discussed in ref.49, such the transverse gauge field fluctuations lead to non-Fermi liquid behavior.

In the continuum theory, or the QED₃ theory, the gauge field associated with fluctuations about the mean field state is either compact or non-compact. By contrast, in our theory the phase fluctuations are compact but the Lagrange multiplier fields are not compact. This may require a different approach to instanton effects discussed in the confinement phenomenon in the compact QED₃ theory.⁵⁰

Acknowledgment

I would like to thank Prof. T. Tohyama for useful discussion. The numerical calculations were carried out in part on Altix3700 BX2 at YITP in Kyoto University.

Appendix: Derivation of the Effective Interaction

In deriving the effective interaction between the fermions, eq.(36), we first integrate over the fermion fields to obtain the effective action for the auxiliary fields, and then the effective interaction between the fermions is derived using the resulting effective action for the auxiliary fields. In this Appendix, we show that this kind of calculation is equivalent to the random phase approximation. To make clear the point, we consider a following simple model,

$$H = \sum_k \xi_k f_k^\dagger f_k + \sum_{k,q} \phi_q f_{k+q}^\dagger f_k + \sum_q K_q \phi_q \phi_{-q}. \quad (\text{A}\cdot 1)$$

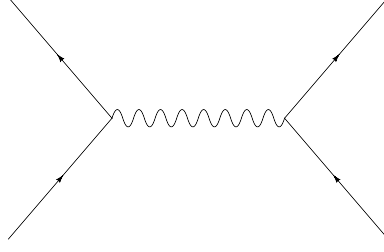


Fig. A.1. The lowest order contribution to the fermion interaction due to the exchange of the ϕ_q field. Wavy line represents the propagator of the ϕ_q field and straight lines represent the fermion propagator.

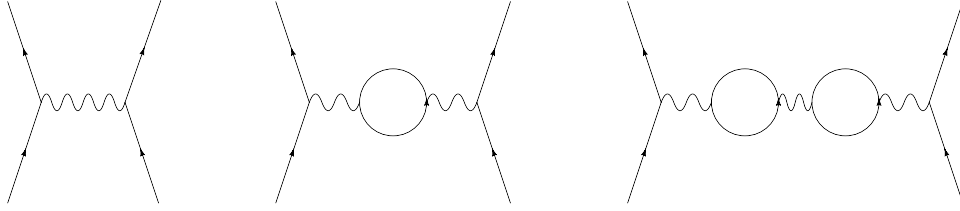


Fig. A.2. Feynman diagram representing the random phase approximation.

The lowest order term for the interaction between fermions mediated by the ϕ_q field is given by

$$V_0 = -\frac{1}{2} \sum_{k,k',q} \frac{1}{K_q} f_{k+q}^\dagger f_{k'}^\dagger f_{k'+q} f_k. \quad (\text{A}\cdot 2)$$

The Feynman diagram for this term is represented by Fig.A.1. By including fermion bubble diagrams and summing over terms represented by Fig.A.2, we obtain

$$V_{RPA} = -\frac{1}{2} \sum_{k,k',q} \frac{1}{2K_q + \Pi_q(i\Omega_n)} f_{k+q}^\dagger f_{k'}^\dagger f_{k'+q} f_k, \quad (\text{A}\cdot 3)$$

where

$$\Pi_q(i\Omega_n) = -\frac{1}{\beta N} \sum_{i\omega_n, k} G_k^{(0)}(i\omega_n) G_{k+q}^{(0)}(i\omega_n + i\Omega_n), \quad (\text{A}\cdot 4)$$

with $G_k(i\omega_n) = 1/(i\omega_n - \xi_k)$.

Now we show that the same interaction term is obtained by the following path-integral calculation. We consider the action for the Hamiltonian (A.1),

$$\begin{aligned} S = & \sum_k f_k^\dagger(i\omega_n) (-i\omega_n + \xi_k) f_k(i\omega_n) + \sum_{k,q} \phi_q(i\Omega_n) f_{k+q}^\dagger(i\omega_n + i\Omega_n) f_k(i\omega_n) \\ & + \sum_q K_q \phi_q(i\Omega_n) \phi_{-q}(-i\Omega_n). \end{aligned} \quad (\text{A}\cdot 5)$$

Integrating out fermion fields, we obtain

$$\begin{aligned}
S_\phi &= -Tr \ln [(-i\omega_n + \xi_k) \delta_{k,k'} \delta_{n,n'} + \phi_{k-k'} (i\omega_n - i\omega_{n'})] + \sum_q K_q \phi_q (i\Omega_n) \phi_{-q} (-i\Omega_n) \\
&= \sum_q \left[K_q + \frac{1}{2} \Pi_q (i\Omega_n) \right] \phi_q (i\Omega_n) \phi_{-q} (-i\Omega_n) + \dots
\end{aligned} \tag{A.6}$$

Using the effective action for the ϕ_q field obtained this way, we consider the following action,

$$\begin{aligned}
S' &= \sum_k f_k^\dagger (i\omega_n) (-i\omega_n + \xi_k) f_k (i\omega_n) + \sum_{k,q} \phi_q (i\Omega_n) f_{k+q}^\dagger (i\omega_n + i\Omega_n) f_k (i\omega_n) \\
&\quad + \sum_q \left[K_q + \frac{1}{2} \Pi_q (i\Omega_n) \right] \phi_q (i\Omega_n) \phi_{-q} (-i\Omega_n).
\end{aligned} \tag{A.7}$$

By integrating out the ϕ_q field, we obtain (A.3).

References

- 1) B. O. Wells, Z. X. Shen, A. Matsuura, D. M. King, M. A. Kastner, M. Greven, and R. J. Birgeneau: Phys. Rev. Lett. **74** (1995) 964.
- 2) S. LaRosa, I. Vobornik, F. Zwick, H. Berger, M. Grioni, G. Margaritondo, R. J. Kelley, M. Onellion, and A. Chubukov: Phys. Rev. B **56** (1997) R525.
- 3) F. Ronning, C. Kim, D. L. Feng, D. S. Marshall, A. G. Loeser, L. L. Miller, J. N. Eckstein, I. Bozovic, and Z. X. Shen: Science **282** (1998) 2067.
- 4) I. Affleck and J. B. Marston: Phys. Rev. B **37** (1988) 3774.
- 5) J. B. Marston and I. Affleck: Phys. Rev. B **39** (1989) 11538.
- 6) T. C. Hsu: Phys. Rev. B **41** (1990) 11379.
- 7) D. H. Kim and P. A. Lee: Ann. Phys. (N. Y.) **272** (1999) 130.
- 8) C. L. Kane, P. A. Lee, and N. Read: Phys. Rev. B **39** (1989) 6880.
- 9) T. Tohyama and S. Maekawa: Supercond. Sci. Technol. **13** (2000) R17.
- 10) D. P. Arovas and A. Auerbach: Phys. Rev. B **38** (1988) 316.
- 11) P. A. Lee, N. Nagaosa, and X. G. Wen: Rev. Mod. Phys. **78** (2006) 17.
- 12) S. Chakravarty, B. I. Halperin, and D. R. Nelson: Phys. Rev. Lett. **60** (1988) 1057.
- 13) N. Read and S. Sachdev: Phys. Rev. B **42** (1990) 4568.
- 14) T. Skyrme: Nucl. Phys **31** (1962) 556.
- 15) T. Morinari: Phys. Rev. B **72** (2005) 104502.
- 16) R. J. Gooding: Phys. Rev. Lett. **66** (1991) 2266.
- 17) F. Waldner: J. Magn. Magn. Mater. **54** (1986) 873.
- 18) S. Belov and B. Kochelaev: Solid State Commun. **106** (1998) 207.
- 19) S. Haas, F.-C. Zhang, F. Mila, and T. M. Rice: Phys. Rev. Lett. **77** (1996) 3021.
- 20) C. Timm and K. H. Bennemann: Phys. Rev. Lett. **84** (2000) 4994.
- 21) E. C. Marino: Phys. Rev. B **61** (2000) 1588.
- 22) A. Moskvin and A. Ovchinnikov: Physica B **259** (1999) 476.
- 23) G. Seibold: Phys. Rev. B **58** (1998) 15520.
- 24) A. R. Moura, A. R. Pereira, and A. S. T. Pires: Phys. Rev. B **75** (2007) 014431.
- 25) G. Baskaran: Phys. Rev. B **68** (2003) 212409.
- 26) F. C. Zhang and T. M. Rice: Phys. Rev. B **37** (1988) 3759.
- 27) A. L. Fetter and J. D. Walecka: *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971)p.498.
- 28) R. Rajaraman: *Solitons and instantons* (North-Holland, 1987)p.115.
- 29) R. D. Pisarski: Phys. Rev. D **29** (1984) 2423.
- 30) T. W. Appelquist, M. Bowick, D. Karabali, and L. C. R. Wijewardhana: Phys. Rev. D **33** (1986) 3704.
- 31) D. Nash: Phys. Rev. Lett. **62** (1989) 3024.
- 32) T. Appelquist, D. Nash, and L. C. R. Wijewardhana: Phys. Rev. Lett. **60** (1988) 2575.
- 33) V. P. Gusynin and M. Reenders: Phys. Rev. D **68** (2003) 025017.
- 34) R. B. Laughlin: J. Phys. Chem. Solids **56** (1995) 1627.
- 35) K. Kondo and H. Nakatani: Prog. Theor. Phys. **87** (1992) 193.
- 36) I. J. R. Aitchison and N. E. Mavromatos: Phys. Rev. B **53** (1996) 9321.

- 37) J. R. Schrieffer, X. G. Wen, and S. C. Zhang: Phys. Rev. B **39** (1989) 11663.
- 38) A. Singh and Z. Tesanovic: Phys. Rev. B **41** (1990) 614.
- 39) H. Chi and A. D. S. Naji: Phys. Rev. B **46** (1992) 8573.
- 40) B. B. Beard, R. J. Birgeneau, M. Greven, and U.-J. Wiese: Phys. Rev. Lett. **80** (1998) 1742.
- 41) J. K. Kim and M. Troyer: Phys. Rev. Lett. **80** (1998) 2705.
- 42) R. R. P. Singh: Phys. Rev. B **39** (1989) 9760.
- 43) R. Coldea, S. M. Hayden, G. Aeppli, T. G. Perring, C. D. Frost, T. E. Mason, S.-W. Cheong, and Z. Fisk: Phys. Rev. Lett. **86** (2001) 5377.
- 44) A. A. Katanin and A. P. Kampf: Phys. Rev. B **66** (2002) 100403(R).
- 45) C. H. Chung, H.-Y. Kee, and Y. B. Kim: Phys. Rev. B **67** (2003) 224405.
- 46) R. Dillenschneider and J. Richert: Phys. Rev. B **73** (2006) 224443.
- 47) T. Pereg-Barnea and M. Franz: Phys. Rev. B **68** (2003) 180506(R).
- 48) X.-G. Wen and P. A. Lee: Phys. Rev. Lett. **76** (1996) 503.
- 49) P. A. Lee and N. Nagaosa: Phys. Rev. B **46** (1992) 5621.
- 50) A. Polyakov: Nucl. Phys. B **120** (1977) 429.